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On maximal codes with a finite interpreting delay

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Abstract

The notion of codes with a finite interpreting delay (f.i.d.) was introduced in (Guesnet, Theoret. Inform. Appl. 34 (2000) 47–59). In this paper, we are interested in the notion of maximality for f.i.d. codes. We characterize the maximal f.i.d. codes in terms of completeness. We also present an embedding procedure keeping thinness, rationality and the delay. © 2002 Elsevier Science B.V. All rights reserved.

Keywords: Maximal codes; Complete sets; Embedding Method; Interpreting delay

1. Introduction

The theory of codes is closely connected with the two notions of completeness and maximality. From this point of view, the equivalence between completeness and maximality has been established for famous families of codes such as thin codes [2, p. 67], thin circular codes [9], thin codes with a finite deciphering delay [3] or thin codes with a finite synchronization delay [5]. As illustrated in [3,5], this question is especially sensitive in the case of codes with delay:

- For any code with a fixed finite deciphering delay d, being maximal is equivalent to being complete.
- However, this equivalence cannot be translated in terms of codes with a fixed synchronization delay d.

When the equivalence holds, it is a natural question to examine the problem of constructing embedding methods. In fact, the existence of an effective procedure for embedding an arbitrary code into a maximal one remains an open problem. From this point of view, given a class of code $\mathscr{F}$ and given $X \in \mathscr{F}$, we are interested in constructing a maximal code $Y \in \mathscr{F}$ that contains X.

- For the family of finite codes, the question remains open. More precisely, finding a procedure to decide whether a finite code has a finite completion is still an open problem [8,13,12].

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- A strong result, due to Schützenberger [14], states that each finite maximal code has either an infinite deciphering delay or a delay equal to 0. Thus, for finite codes with a finite deciphering delay, one can embed only the codes with delay 0.

In fact, a procedure of completion has been obtained for thin codes [10], thin prefix codes, rational bifix codes [15,6], thin circular codes [1], thin codes with a fixed finite deciphering delay [7] and uniformly synchronous codes [5] (but without preserving the delay).

In this paper, we deal with codes with a finite-interpreting delay (f.i.d.). Informally, if X is a code with a finite interpreting delay, then any "long enough" word w in X^* has a unique interpretation (taking account of adjacent interpretations). More precisely, we define the *interpreting delay* of X as the smallest integer n such that $\beta X^*\alpha \cap X^n = \emptyset$, for all pairs of words $(\alpha, \beta) \notin X^* \times X^*$, with α prefix of a word of X, β a suffix of a word of X. A remarkable property of these codes is the characterization of the class of finite f.i.d. codes as the intersection between the class of finite circular codes and the class of adjacent codes. This leads to a corresponding version of the famous defect theorem [11].

Our study consists in establishing the two main results which follows.

- First, we prove that the following property holds:

Theorem. *Let X be a thin f.i.d. code with delay d. The following three properties are equivalent*:

(i) *X is complete.*
(ii) *X is a maximal code.*
(iii) *X is maximal in the family of f.i.d. codes with delay d.*

It is of interest to note that maximal f.i.d. codes are necessarily infinite. This fact is more precisely investigated in the second part of our paper.

- Secondly, we consider the problem of embedding a given f.i.d. code in a maximal one. Given an arbitrary thin f.i.d. code X, we present an effective construction of a maximal thin f.i.d. code containing X. Moreover, our construction preserves the regularity of the sets. It is important to mention that our procedure embeds a code with a finite interpreting delay in a complete one with the same delay. This is not possible in the case of codes with a finite synchronization delay [5].

We now describe the content of our paper. In Section 2, we recall some basic notions and some well-known results on codes. In Section 3, we define the codes with a f.i.d. We also give some relations between codes with a f.i.d. and uniformly synchronous codes. In Section 4, we prove that any thin code with a f.i.d. is complete iff it is maximal. Finally, in the last section we give an embedding procedure for thin codes with a f.i.d.

2. Preliminaries

We denote by A an alphabet, by A^* the free monoid it generates and by ε the empty word. We suppose in the following that $|A| > 1$.

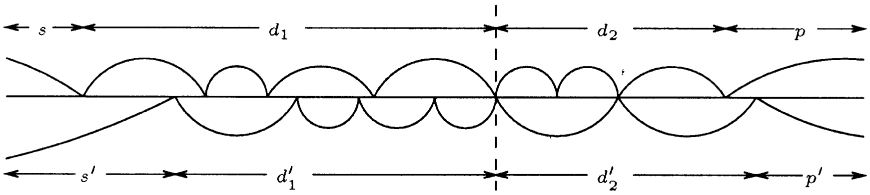

Fig. 1. Two adjacent interpretations.

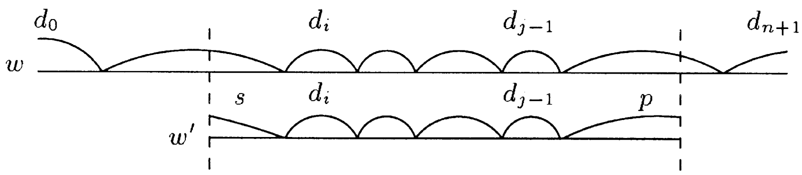

Fig. 2. Induced interpretation.

Given a word $w \in \Sigma^*$, the set of all factors (prefixes, suffixes) of w is denoted by $F(w)$ $(P(w), S(w))$. We denote by $|w|$ the length of the word w. A word w is *unbordered* if no proper non-empty prefix of w is a suffix of w.

Let $X \subset A^*$ and let $w \in A^*$. An X-interpretation of w is a triple (s, d, p) such that $s \in S(X)$, $d \in X^*$, $p \in P(X)$ and $w = sdp$. Two X-interpretations (s, d, p) and (s', d', p') are *adjacent* if there exist $d_1, d_2, d_1', d_2' \in X^*$ such that $d = d_1 d_2$, $d' = d_1' d_2'$, $sd_1 = s'd_1'$ and $d_2 p = d_2' p'$ (see Fig. 1).

Let $w \in X^*$, the X-interpretation $(\varepsilon, w, \varepsilon)$ is the *trivial interpretation* of w. An X-interpretation (s, d, p) such that $s, p \in X^*$ is *quasi-trivial.*

Let $w \in A^*$ and let $u, w', v \in A^*$ such that $w = uw'v$. Let $(d_0, d_1, \ldots, d_n, d_{n+1})$ be an X-interpretation of w with $d_1, \ldots, d_n \in X$.

The X-interpretation $(d_0, d_1, \ldots, d_n, d_{n+1})$ *induces* an X-interpretation for w' iff there exist $s \in S(X)$, $p \in P(X)$ and $i, j \in \mathbb{N}$, $0 < i \leqslant j \leqslant n+1$ such that $s \in S(d_{i-1})$, $p \in P(d_j)$ and $w' = sd_i d_{i+1} \ldots d_{j-1} p$ (see Fig. 2). The X-interpretation $(s, d_i, \ldots, d_{j-1}, p)$ is the X-interpretation of w' *induced* by $(d_0, d_1, \ldots, d_n, d_{n+1})$.

We will recall now the definitions of some well-known codes:

A non-empty subset $X \subset A^+$ is a code if for any $x_1, \ldots, x_n, y_1, \ldots, y_m \in X$ the following condition holds:

$$x_1 \ldots x_n = y_1 \ldots y_m \quad \Rightarrow \quad n = m, \ x_i = y_i, \quad i \in [1, n].$$

A *thin* code X is a code such that $A^* \backslash F(X) \neq \emptyset$. A *very thin* code X is a code satisfying $X^* \cap (A^* \backslash F(X)) \neq \emptyset$. That is to say, there exists a word in X^* which cannot be extended into a word of X. A code is *maximal* if it is not strictly included in another one. A code X is *complete* if $F(X^*) = A^*$.

A code X is *synchronous* if there exist $x, y \in X^*$ such that for all words $u, v \in A^*$,

$$uxyv \in X^* \quad \Rightarrow \quad ux, yv \in X^*.$$

A code X is *uniformly synchronous* if there exists $\sigma \in \mathbb{N}$ such that for all $x, y \in X^\sigma$ and $u, v \in A^*$,

$$uxyv \in X^* \quad \Rightarrow \quad ux, yv \in X^*. \tag{1}$$

The smallest integer σ satisfying (1) is the *synchronization delay* of X.

3. Codes with a finite interpreting delay

Let X be a code. This paper deals with the following two notions of codes introduced in [11]:

- X has a finite interpreting delay if there exists $m \geqslant 1$ such that for all $\alpha \in P(X)$, $\beta \in S(X)$, $(\alpha, \beta) \notin X^* \times X^*$, we have

  $$\beta X^* \alpha \cap X^m = \emptyset. \tag{2}$$

  The *interpreting delay* is the smallest integer m satisfying Condition (2). Thus, if X is a code with a finite interpreting delay m, any X-interpretation of a word in $X^m \cdot X^*$ is quasi-trivial. In the following, such a code will be called f.i.d. code for short.
- X is an adjacent code if $X \cap (S(X) \backslash \{\varepsilon\}) \cdot X^+ = \emptyset$ and $X \cap X^+ \cdot (P(X) \backslash \{\varepsilon\}) = \emptyset$. Note that, if X is an adjacent code, we have $X^+ \cap (S(X) \backslash X^*) \cdot X^+ = \emptyset$ and $X^+ \cap X^+ \cdot (P(X) \backslash X^*) = \emptyset$.

  Clearly, any f.i.d. code is an adjacent code.

The definition of f.i.d. codes implies a nice property, concerning adjacent interpretations. In fact, if a word has two adjacent interpretations then these are "everywhere" adjacent:

Proposition 3.1. *Let X be a f.i.d. code and let $x \in X^+$. Any interpretation (s, d, p) which is adjacent to the trivial one satisfies $s, p \in X^*$. Moreover, if $x \in X$ then (s, d, p) is equal to one of the following three interpretations*: $(\varepsilon, \varepsilon, x)$, $(\varepsilon, x, \varepsilon)$ *and* $(x, \varepsilon, \varepsilon)$.

Proof. The interpretation (s, d, p) of x is adjacent to the trivial one, thus there exist $x_1, x_2, d_1, d_2 \in X^*$ such that $x = x_1 x_2$, $d = d_1 d_2$ and $sd_1 = x_1$, $d_2 p = x_2$. The triple (s, d_1, ε) is an interpretation of x_1. Since X is a f.i.d. code, it is an adjacent code, thus $s \in X^*$. In a similar way $p \in X^*$.

If $x \in X$ then, since X is a code, $x = sdp$ with $s, d, p \in X^*$ implying that two of these words are equal to ε and the last is equal to x. This completes the proof. □

To end this section we present two propositions which link f.i.d. codes with uniformly synchronous codes.

Proposition 3.2. *Let X be a f.i.d. code. Then the following two conditions are equivalent*:

(i) *There exists $n \in \mathbb{N}$ such that*

$$X \cap A^*X^nA^* = \emptyset. \tag{3}$$

(ii) *X is a uniformly synchronous code.*

It is interesting to note that Condition (3) is the one that rational circular codes must satisfy in order to be uniformly synchronous [13].

Proof of Proposition 3.2. Let X be a f.i.d. code with delay δ and let $n \in \mathbb{N}$ such that $X \cap A^*X^nA^* = \emptyset$. We shall prove that X is a uniformly synchronous code.

Let x, y be two words of X^m where $m = \max(n, \delta)$. Let $u, v \in A^*$ such that $uxyv \in X^*$. Hence, there exist $d_0, d_1, \ldots, d_k \in X$, $k \geqslant 0$ such that $uxyv = d_0d_1 \ldots d_k$. Let $d' \in A^+$, $d'' \in A^*$ and let $i \in \mathbb{N}$ such that $d_0 \ldots d_{i-1}d' = ux$, $d''d_{i+1} \ldots d_k = yv$ and $d_i = d'd''$. Since $m = \max(n, \delta)$, we have $x \in X^n \cdot X^*$. Thus, we have $x \notin F(x)$ ($X \cap A^*X^nA^* = \emptyset$). Moreover, we have $d' \in P(d_i)$, hence $|d'| < |x|$. Therefore, the X-interpretation $(\varepsilon, d_0 \ldots d_{i-1}, d')$ induces an X-interpretation for the suffix x of ux. As $x \in X^{\delta} \cdot X^*$ and X has an interpreting delay δ, this interpretation is quasi-trivial, that is $d' \in X^*$.

In a similar way, we have $d'' \in X^*$. Hence, $ux \in X^*$ and $yv \in X^*$. Consequently, X is a uniformly synchronous code (its delay is lower than or equal to m).

Conversely, if X is a uniformly synchronous code with delay σ, then Eq. (3) is trivially satisfied for $n = 2\sigma$. $\square$

The following two examples show that Eq. (3) is not satisfied by any f.i.d. code.

Example 3.1. The set $b + ab^*c$ is a f.i.d. code with delay 1. However it is not uniformly synchronous [5].

Example 3.2. Another interesting example of a f.i.d. code which is not uniformly synchronous is the restricted Dyck code D_1'. Indeed, it is composed of the words $w \in A^*$ such that $|w|_a = |w|_b$ and such that for each proper prefix $u \neq \varepsilon$ of w, we have $|u|_a > |u|_b$. This code is dense [9], that is it is not thin. Therefore, it is not a uniformly synchronous code [5]. However, it is a f.i.d. code with delay 1. Indeed, let (s, d, p) be a D_1'-interpretation of a word w belonging to D_1'. By the definition of an interpretation, s is a suffix of a word in D_1', hence for any suffix u of s, we have $|u|_b \geqslant |u|_a$. Moreover, s is a prefix of w, hence for any prefix v of s, we have $|v|_a \geqslant |v|_b$. Therefore, $|s|_a = |s|_b$ and we even have $s \in D_1'^*$. In a similar way, we have $p \in D_1'^*$. Thus, the code D_1' is a f.i.d. code with delay 1.

Proposition 3.3. *Let X be a uniformly synchronous code. The following two conditions are equivalent*:

(i) *X is adjacent.*

(ii) *X is a f.i.d. code.*

Proof. Any f.i.d. code is adjacent. We must now prove that uniformly synchronous code X which is adjacent is a f.i.d. code.

Let $\sigma\in\mathbb{N}$ be the delay of synchronization of X. Let $x\in X^{2\sigma}$ and let $s\in S(X)$, $p\in P(X)$, $y\in X^*$ such that $x=syp$. Without loss of generality, we can assume that $s\notin X$ and $p\notin X$ (for example, if $s\in X$ then we consider the interpretation (ε, sy, p) instead of (s, y, p)).

By definition of x and y, there exist $x_1,\dots,x_{2\sigma}\in X$ and $k\geqslant 0$, $y_1,\dots,y_k\in X$ such that $x=x_1\dots x_{2\sigma}$ and $y=y_1\dots y_k$. We set $f=x_1\dots x_\sigma$ and $g=x_{\sigma+1}\dots x_{2\sigma}$.

There exists $u\in A^+$ such that $us\in X$ (since $s\in S(X)\backslash X$). In a similar way, as $p\in P(X)\backslash X$, there exists $v\in A^+$ such that $pv\in X$. We have $ufgv=us\cdot y_1\dots y_k\cdot pv$ thus $ufgv\in X^*$. Since X has synchronization delay σ and $f,g\in X^\sigma$, we have $uf, gv\in X^*$. The set X is a code and we have $uf\cdot gv=us\cdot y_1\dots y_k\cdot pv$, thus there exist $j\in\mathbb{N}$ such that $uf=usy_1\dots y_j$, that is, $f=sy_1\dots y_j$. As the set X is adjacent, we have $s\in X^*$. In a similar way, we have $p\in X^*$. Therefore the set X is a f.i.d. code with delay lower than or equal to 2σ. $\square$

Example 3.3. The set $\{ba, bad, db\}$ is a uniformly synchronous code but it is not a f.i.d. code (it is not adjacent).

4. Maximal f.i.d. codes

Any code is contained in a maximal code [2, p. 41]. This is always true for codes with a finite deciphering delay [4]. We shall prove the same result for f.i.d. codes. The proof is done classically by applying the Zorn lemma.

Proposition 4.1. *Any f.i.d. code with delay d is contained in some code which is maximal in the family of f.i.d. codes with delay d.*

Proof. Let X be a f.i.d. code with delay d and let $\mathscr{F}$ be the set, ordered by inclusion, of the f.i.d. codes with delay d containing X. Let $\mathscr{C}$ be a totally ordered chain of $\mathscr{F}$. Let $Y=\bigcup_{Z\in\mathscr{C}} Z$.

We shall prove that Y is an upper bound for $\mathscr{C}$ i.e., Y is a f.i.d. code with delay d. We know that Y is a code (see e.g. [2, pp. 41, 42]).

It must now be proved that Y has a finite interpreting delay. Let $m\geqslant 0$, $\alpha\in P(Y)$, $\beta\in S(Y)$, $x_1,x_2,\dots,x_d\in Y$ and $y_1,y_2,\dots,y_m\in Y$ such that $x_1x_2\dots x_d=\beta y_1y_2\dots y_m\alpha$. By definition of Y, each x_i $(1\leqslant i\leqslant d)$ and y_j $(1\leqslant j\leqslant m)$ belongs to an element of $\mathscr{C}$. In a similar way, α (β) is a prefix (suffix) of an element of $\mathscr{C}$. Since the chain is ordered by inclusion, there exists a f.i.d. code Z' in the chain that contains the preceding elements of $\mathscr{C}$. The f.i.d. code Z' has delay d thus $\alpha,\beta\in Z'^*$. Therefore, $\alpha,\beta\in Y^*$ and Y is a f.i.d. code with delay d.

By Zorn lemma, we obtain that $\mathscr{F}$ has a maximal element. $\square$

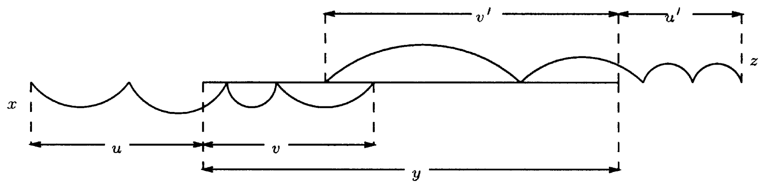

Fig. 3. The word $\alpha_X(y)$.

The following is devoted to the proof of the equivalence between the notions of completeness and maximality for thin f.i.d. codes. We know that this is the case for circular codes [2] and codes with a finite deciphering delay [3].

Theorem 4.2. *Given a thin f.i.d. code X with delay d, the following three properties are equivalent*:
(i) *X is complete.*
(ii) *X is a maximal code.*
(iii) *X is maximal in the family of f.i.d codes with delay d.*

We begin by introducing the following notation.

Let y be an unbordered word and X be a f.i.d code with delay n. Let X' be the set of the words in X^n which have a (non-empty) suffix which is a prefix of y. If $X' = \emptyset$, we set $u = \varepsilon$, otherwise there exists a word $x \in X'$ and some words u, v, y' such that $x = uv, y = vy'$ and v of maximal length (such a word exists since $|y|$ is finite). In a similar way, let X'' be the set of words in X^n which have a (non-empty) prefix which is a suffix of y. If $X'' = \emptyset$, we set $u' = \varepsilon$, otherwise there exists a word $z \in X''$ and some words u', v', y'' such that $z = v'u'$, $y = y''v'$ and v' of maximal length.

We denote by $\alpha_X(y)$ the word uyu' (see Fig. 3).

The proof of Theorem 4.2 is based upon the following result:

Proposition 4.3. *Let X be a f.i.d. code with delay n and let y an unbordered word such that $y \notin F(X^*)$. Then the code $Y = X \cup \{\alpha_X(y)\}$ is a f.i.d. code with delay n.*

Proof. Let $t = \alpha_X(y)$.

First, we shall prove that t has no non-trivial Y-interpretation.

Assume that t has a non-trivial Y-interpretation (s, d, p). Of course, t is not a factor of d (since the interpretation is non-trivial, we have $|d| < |t|$), thus $d \in X^*$. Let (s', d', p') be the interpretation of y induced by (s, d, p). It is clear that such induced interpretation exists since y is not a factor of a word in X. As $d \in X^*$, we have $d' \in X^*$. We shall prove that $p' \in P(X^*)$.

If $p' \in P(t)$ then, since y is unbordered and since $|p'| < |y|$, we have $p' \in P(u)$ (we recall that $t = uyu'$, Fig. 4). By definition of t, we have $u \in P(X^n)$, thus $p' \in P(X^n)$.

Therefore, we have $p' \in P(Y)$ with $Y = X \cup \{t\}$, thus $p' \in P(X)$ or $p' \in P(X^n)$. Hence $p' \in P(X^*)$.

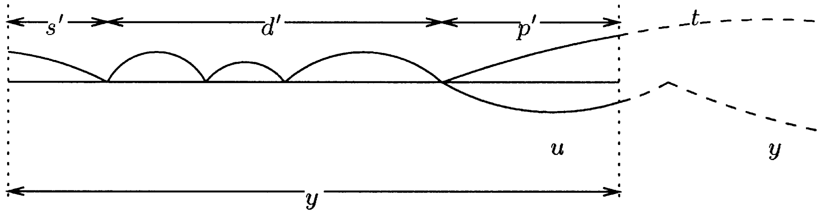

Fig. 4. Interpretation of y with $p' \in P(t)$.

In a similar way, we have $s' \in S(X^*)$.

We have $y = s'd'p'$, thus $y \in F(X^*)$ which is in contradiction with the definition of y. Therefore, t has no non-trivial Y-interpretation.

Now, we can prove that Y is a code.

Assume that there exist $x_1, x_2, \ldots, x_r \in Y$, $y_1, y_2, \ldots, y_m \in Y$ with $r, m \geqslant 1$ such that $x_1 x_2 \ldots x_r = y_1 y_2 \ldots y_m$.

Since X is a code, we cannot have $x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_m \in X$. We can assume without loss of generality that there exists i such that $x_i = t$.

In this case, since t has no non-trivial Y-interpretation, the Y-interpretation $(\varepsilon, y_1 y_2, \ldots, y_m, \varepsilon)$ of $x_1 x_2 \ldots x_r$ cannot induce a Y-interpretation for x_i. Therefore, there exists k, $1 \leqslant k \leqslant m$ such that x_i is a factor of y_k. Since $t \notin F(X)$, we have $x_i = y_k = t$ and $y_1 \ldots y_{k-1} = x_1 \ldots x_{i-1}$, $x_i = y_k = t$, $y_{k+1} \ldots y_m = x_{i+1} \ldots x_r$. By iterating this process to all words t appearing in the equation, we obtain $r = m$ and $x_i = y_i$ for $1 \leqslant i \leqslant r$.

Thus Y is a code.

Finally, assume that a word $w \in Y^n$ has a Y-interpretation (s, d, p). We shall prove that this interpretation is quasi-trivial. Let $w_1, \ldots, w_n \in Y$ and let $d_1, \ldots, d_m \in Y$ such that $w = w_1 \ldots w_n$ and $d = d_1 \ldots d_m$.

- Assume first that $w, d \in X^*$.
  We recall that $t = uyu'$ and $uv \in X^n$, $v'u' \in X^n$. We shall prove that $p \in P(X^n)$.
  If $p \in P(t)$ then $|p| < |u| + |y|$ since $y \notin F(X^*)$ and p is a suffix of $w \in X^n$. Moreover, $|v|$ is the maximal length of the overlaps between the words of X^n and y, thus $|p| \leqslant |u| + |v|$. Therefore, as $uv \in X^n$ is a prefix of t, we have $p \in P(uv)$ hence $p \in P(X^n)$.
  In a similar way, we have $s \in S(X^n)$. Thus, the interpretation (s, d, p) is an X-interpretation of a word in X^n. Therefore, $s \in X^*$ and $p \in X^*$. Since $X^* \subset Y^*$, we have $s, p \in Y^*$.
- Assume now that t is a Y-factor of d.
  Since t has no non-trivial Y-interpretation and $t \notin F(X)$, t is a Y-factor of w and there exist $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$ such that

$$w_1 \ldots w_{i-1} = sd_1 \ldots d_{j-1}, \quad w_i = d_j = t, \quad w_{i+1} \ldots w_n = d_{j+1} \ldots d_m p. \tag{4}$$

We consider the smallest integer i_0 such that $w_{i_0} = t$. Let j_0 be such that $w_1 \ldots w_{i_0-1} = sd_1 \ldots d_{j_0-1}$. We have $w_1, \ldots, w_{i_0-1}, d_1, \ldots, d_{j_0-1} \in X$.

If $s \notin Y^*$ then, by (4), we have $i_0 > 1$ (since $i_0 = 0$ yields $s = \varepsilon$). Moreover, as we have $w_1 \ldots w_{i_0-1} = sd_1 \ldots d_{j_0-1}$ with $w_1 \ldots w_{i_0-1} \in X^*$, the word s is a prefix of a word in X^n. By definition of t, if $s \in S(t)$ then $s \in S(X^n)$. Since X is a f.i.d. code, it is an adjacent code, therefore $w_1 \ldots w_{i_0-1} = sd_1 \ldots d_{j_0-1}$ yields $s \in X^*$ which contradicts $s \notin Y^*$.

Thus $s \in Y^*$. Symmetrically, we have $p \in Y^*$.

In a similar way, if t is a Y-factor of w then there exist i, j, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$ such that (4) holds, hence $s, p \in Y^*$.

Consequently, (s, d, p) is a quasi-trivial interpretation. Hence, the code Y is a f.i.d. code with delay n. □

We can give the Proof of Theorem 4.2.

Proof of Theorem 4.2. If X is a thin complete set then it is maximal [2]. Moreover, if X is maximal then it is maximal in the family of f.i.d. codes of delay d.

We must now prove that if X is maximal in the family of f.i.d. codes with delay d then it is complete. Assume that X is not complete, then Proposition 4.3 implies that X is not maximal since we can construct an unbordered, uncompletable word with any uncompletable word [2, p. 10] (we have supposed $|A| > 1$). □

Remark 4.1. The set Y that was constructed in Proposition 4.3 is very thin. Indeed, since y is a factor of t and $y \notin F(X)$, the word $t^2 \in Y^*$ is not a factor of a word in X. Hence, $t^2 \notin F(X \cup t)$.

5. Completion of f.i.d. codes

First, we show with an example that the Ehrenfeucht–Rozenberg Algorithm [10] cannot be applied to the completion of adjacent codes (and thus, a fortiori to f.i.d. codes).

Example 5.1. Let $A = \{a, b\}$. Clearly, the set $X = a + ab^+a$ is adjacent (this is a f.i.d. code with delay 1). It is not complete since the word $abab$ is uncompletable. However, for any unbordered word y, the set $Y = X \cup \{y\}$ is no longer adjacent. Indeed, as y is unbordered, its first and last letters are different. If its first letter (last letter) is a b then we have $y = b^n au$ ($y = uab^n$) with $n \geqslant 1$ and $u \in \{a, b\}^+$, then there is a non quasi-trivial Y-interpretation $(\varepsilon, a, b^n a)$ ((ab^n, a, ε)) for the word $ab^n a$. As the Ehrenfeucht–Rozenberg Algorithm adds an unbordered word to the set to be completed, if we apply this algorithm we do not obtain an adjacent code.

Moreover, from Proposition 4.3, we could assume that using $\alpha_X(y)$ instead of an unbordered word y in the Ehrenfeucht–Rozenberg Algorithm should lead to a complete f.i.d. code. Unfortunately, this is not true.

Indeed, with the previous code X, the word $w=(ababa)^2(ba)$ belongs to the set

$$Y = \alpha_X(abab)(U\alpha_X(abab))^*,$$

where $U=A^*-X^*-A^*\alpha_X(abab)A^*$ and $\alpha_X(abab)=ababa$.

However, $(\varepsilon, \alpha_X(abab), \alpha_X(abab)ba)$ is a Y-interpretation for w.

The following proposition exhibits a property that f.i.d. codes must satisfy, in order to be complete. This proposition gives an idea about a possible completion method for f.i.d. codes.

Proposition 5.1. *Given a complete very thin f.i.d. code X, the following property holds*:

$$\exists x, y \in X^*, \quad xA^*y \subset X^*. \tag{5}$$

Proof. The Proposition 6.5 in [2, p. 240] states the equivalence between complete synchronous codes and codes which satisfy (5) (note that the hypothesis “X is very thin” in [2] is not necessarily true). It must now be proved that very thin f.i.d. codes are synchronous.

Let d be the delay of X. Let $t_0 \in X^*\backslash F(X)$. Any pair (t_0x, t_0x), where $x\in X^{d-2}$ is a synchronizing pair. Indeed, for any words u,v, if $ut_0xt_0xv\in X^*$ then, since $t_0\notin F(X)$, we have two interpretations for $t_0x\in X^d\cdot X^*$, one induced by $ut_0x\in P(X^*)$ and the other induced by $t_0xv\in S(X^*)$. Let (s,y,p) be the interpretation of t_0x induced by ut_0x and let s' be the word such that $s'syp=ut_0x$. Note that we have $s's\in X^*$. As $t_0x\in X^d\cdot X^*$, (s,y,p) is a quasi-trivial interpretation, that is, $p\in X^*$. Hence, $s's\cdot y\cdot p\in X^*$. Thus $ut_0x\in X^*$. Symmetrically, we have $t_0xv\in X^*$. □

Property (5) is close to the one which is proved in [5] for uniformly synchronous codes. From this point of view, the procedure of embedding f.i.d. codes in maximal ones uses the same process as for uniformly synchronous codes. The main difference is the use of only one marker instead of all the words in $X^{2\sigma}$ for uniformly synchronous codes (where σ stands for the delay of these codes). We will construct two words t_1 and t_2 from a word $t_0\in X^*\backslash F(X)$ in order to consider the words of $t_1A^*t_2$ (as in [10]) instead of the words in $t_0A^*\cap A^*t_0$ (as in [5]). It is interesting to see that this embedding procedure keeps the same delay, while this is not possible for uniformly synchronous code.

Theorem 5.2. *Let X be a very thin f.i.d. code with delay d. Let $t_0\in X^d\cdot X^+\backslash F(X)$. Let a,b be two different letters of A^* and let $t_1=t_0a^{|t_0|}$, $t_2=b^{|t_0|}t_0$. Then the set*

$$X\cup(t_1A^*t_2 - A^*t_2X^*t_1A^* - A^*t_2X^+ - X^+t_1A^* - X^*)$$

is a complete very thin code with the same interpreting delay as X.

In the following, we consider a very thin f.i.d. code X with delay d, $t_0\in X^d\cdot X^+\backslash F(X)$, where $t_0=t_{0,0}t_{0,1},\ldots,t_{0,|t_0|_X-1}$, with $t_{0,i}\in X$ for $0\leqslant i\leqslant |t_0|_X-1$, a,b are two

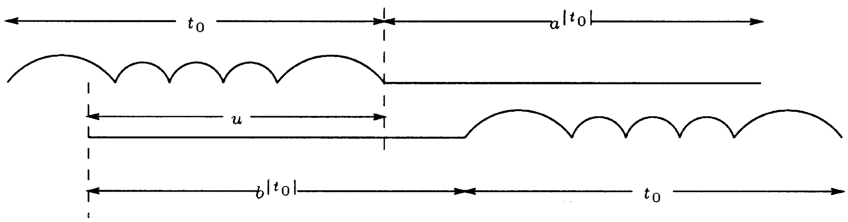

Fig. 5. $t \in S(t_1) \cap P(t_2)$ where $|t_0| < |t| < |t_1|$.

different letters of A^* and $t_1 = t_0 a^{|t_0|}$, $t_2 = b^{|t_0|} t_0$. We set

$$C = t_1 A^* t_2 - A^* t_2 X^* t_1 A^* - A^* t_2 X^+ - X^+ t_1 A^* - X^*$$

and

$$Y = X \cup C.$$

Proposition 5.1 does not give an equivalence between complete very thin codes and Property (5). So to prove Theorem 5.2 we must establish that Y is a code, that Y has delay d, that Y is complete and finally that Y is very thin. This is the aim of Propositions 5.6–5.9.

In order to simplify the proof of the theorem, we start with two lemmas:

Lemma 5.3. *With the preceding notation, the words t_1 and t_2 satisfy the following three properties*:

(i) $S(t_1) \cap P(t_2) = \{\varepsilon\}$.

(ii) $\forall u \in S(t_2),\ x \in P(X^* t_1),\ t_1 = ux \Rightarrow u \in X^*$.

(iii) $\forall v \in P(t_1),\ x \in S(t_2 X^*),\ t_2 = xv \Rightarrow v \in X^*$.

Proof. First, we shall prove the property (i):

Let $t \in S(t_1) \cap P(t_2)$.

- If $|t| \leqslant |t_0|$ then, since by definition we have $t_1 = t_0 a^{|t_0|}$, there exists $k \in \mathbb{N}$ such that $t = a^k$. In a similar way, since $t_2 = b^{|t_0|} t_0$ we have $t = b^k$. Hence, since $a \neq b$, we have $k = 0$, thus $t = \varepsilon$.
- If $|t_0| < |t| < |t_1|$ (see Fig. 5), there exists $u \in S(t_0)$ such that $t = u a^{|t_0|}$ with $|u| < |t_0|$ (since $t \in S(t_1)$). In a similar way, there exists $v \in P(t_0)$ such that $t = b^{|t_0|} v$ with $|v| < |t_0|$. We have $t = u a^{|t_0|} = b^{|t_0|} v$, i.e., $u^{-1} t = a^{|t_0|} = b^{|t_0| - |u|} v$. We have $|t_0| - |u| > 0$, thus, since $a \neq b$, this case does not appear.
- If $|t| = |t_1|$ then, since $t \in S(t_1) \cap P(t_2)$, we have $t = t_1 = t_2$ and thus $t_0 = b^{|t_0|} = a^{|t_0|}$. This case is impossible.

Therefore, if $t \in S(t_1) \cap P(t_2)$ then $t = \varepsilon$.

We prove the second property. Let $u \in S(t_2)$ and let $x \in P(X^* t_1)$ such that $t_1 = ux$.

- If $|u| > |t_0|$ then, since $u \in S(t_2)$, we have $u = b^k t_0$ with $k > 0$. In this case, since $t_1 = ux$, we have $t_0 a^{|t_0|} = b^k t_0 x$, thus $t_0 a^k = b^k t_0$. Therefore, b^k and a^k are period

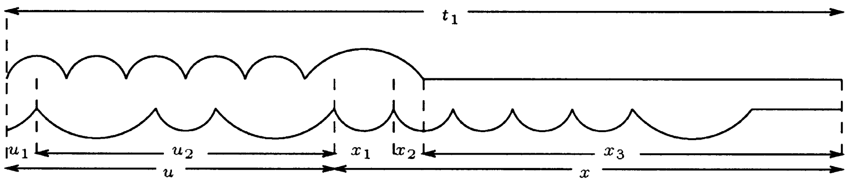

Fig. 6. $t_1 = ux$ with $|u| \leqslant |t_0|$.

words of t_0, thus $t_0 \in b^+$ and $t_0 \in a^+$. We have $a \neq b$, therefore this case cannot appear.

- If $|u| \leqslant |t_0|$ then $u \in S(X^*)$ since $u \in S(t_2)$ and, by definition, $t_2 = b^{|t_0|}t_0$ with $t_0 \in X^*$. Therefore, since $t_1 = ux$ with $t_1 = t_0 a^{|t_0|}$, the word ux induces an X-interpretation (u_1, u_2x_1, x_2) for t_0 with $u = u_1u_2$, $u_1 \in S(X)$, $u_2 \in X^*$ and $x = x_1x_2x_3$, $x_1 \in X^*$, $x_2 \in P(X)$, $x_3 \in A^*$ (see Fig. 6). This is quasi-trivial since X is a f.i.d. code with delay d and $t_0 \in X^d \cdot X^+$. Therefore, $u_1 \in X^*$, hence $u \in X^*$.

The third property can be proved in a similar way. □

Lemma 5.4. *Let $c \in C$. The Y-interpretations of c are $(\varepsilon, c, \varepsilon)$, $(c, \varepsilon, \varepsilon)$ and $(\varepsilon, \varepsilon, c)$.*

Proof. Let (s, y, p) be a Y-interpretation of $c \in C$, different from $(\varepsilon, c, \varepsilon)$, $(c, \varepsilon, \varepsilon)$ and $(\varepsilon, \varepsilon, c)$.

Assume first that $s \in S(C)$. We shall prove that $s \in X^*$.

- If $|s| \leqslant |t_2|$ then $s \in S(t_2)$ (since t_2 is a suffix of any word in C). Thus, since t_1 is a prefix of c, there exists $x \in P(X^*t_1)$ such that $t_1 = sx$ (see Fig. 7, x is a prefix of yp). From Lemma 5.3 we have $s \in X^*$.
- If $|s| > |t_2|$, then there exists $s' \in A^*$ such that $s = s't_2$. We have $c = s't_2yp$. We prove first that $y \in X^*$. Assume the contrary. Then, since t_1 is a prefix of any word in C, we have $y \in X^*t_1A^*$. Thus $c \in A^*t_2X^*t_1A^*$. This is impossible by definition of C. Hence $y \in X^*$. Moreover, by definition of C, we have $p \notin X^+$ (since $c \notin A^*t_2X^+$). If $p = \varepsilon$ then, since we have assumed that (s, y, p) was different from $(c, \varepsilon, \varepsilon)$, we have $y \in X^+$, thus implying $c \in A^*t_2X^+$ which contradicts $c \in C$. Therefore we have $p \notin X^*$.
  - If $p \in P(X)$, since $t_0 \notin F(X)$ and since t_0 is a suffix of c, we have $|p| < |t_0|$. Then (s, y, p) induces an X-interpretation (s'', y', p) for the suffix t_0 of c ($s'' \in S(X)$ since $s \in S(b^{|t_0|}t_0)$ and $|s'| < |t_0|$). Since the code X has delay d and $t_0 \in X^d \cdot X^+$, this is quasi-trivial, i.e. $p \in X^*$ and we have seen that this case cannot appear.
  - Therefore we have $p \in P(C)$. If $|p| \leqslant |t_0|$, then $p \in P(X^*)$ and the interpretation (s, y, p) induces an X-interpretation for t_0, it is quasi-trivial and thus $p \in X^*$; we have seen that this case cannot appear. Thus we have $|p| > |t_0|$, which implies that $|p| \geqslant |t_2|$ (otherwise $t_2 = x'p$ with $x' \in S(t_2X^*)$ and we have, with Lemma 5.3(iii), $p \in X^*$). Thus $c \in A^*t_2X^*t_1A^*$, this is in contradiction with the definition of C.

Therefore we do not have $|s| > |t_2|$.

We have proved that if $s\in S(C)$ then $s\in X^*$.

Assume now that $s\in S(X)$. Since $t_0\notin F(X)$, we have $|s|<|t_0|$. But in this case (s,y,p) induces an X-interpretation for the prefix t_0 of c. This interpretation is quasi-trivial, hence $s\in X^*$.

Therefore we have $s\in X^*$. In a similar way, we have $p\in X^*$. Since $c\notin X^*$, we have $y\notin X^*$. However, if $s\neq\varepsilon$ ($p\neq\varepsilon$), then $c\in X^+t_1A^*$ ($c\in A^*t_2X^+$). This is in contradiction with the definition of C. Therefore, we have $s=p=\varepsilon$. This invalidates the hypothesis that (s,d,p) is different from $(\varepsilon,c,\varepsilon)$. □

Corollary 5.5. *The words of C have no X-interpretations.*

Proof. Trivially, if $c\in C$ has an X-interpretation, then either $s=c$ (or $p=c$) which is in contradiction with $t_0\notin F(X)$ or $d=c$ which is in contradiction with the definition of C (we have $C\cap X^*=\emptyset$). □

Proposition 5.6. *The set Y is a code.*

Proof. Let $y_0,y_1,\ldots,y_n\in Y$ and let $y'_0,y'_1,\ldots,y'_m\in Y$ such that $y_0y_1\ldots y_n=y'_0y'_1\ldots y'_m$ with $n,m\geqslant 0$.

Without loss of generality, we can assume that $|y_0|>|y'_0|>0$. Thus, the word y_0 has a Y-interpretation $(\varepsilon,y'_0y'_1\ldots y'_k,z)$ with $k\geqslant 0$ and $y_{k+1}=zz'$, where $z'\in A^*$ if $k+1\leqslant m$ and $z=z'=\varepsilon$ otherwise.

- Assume first that $y_0\in X$. Since $t_0\notin F(X)$, we have $y'_i\in X$ with $0\leqslant i\leqslant k$ (t_0 is a prefix and a suffix of any word in C). Moreover, in a similar way, $z\in P(X^*)$. We denote by $z''\in X^*$ and by $p\in P(X)$ some words such that $z=z''p$. Then $(\varepsilon,y'_0y'_1\ldots y'_kz'',p)$ is an X-interpretation for y_0. The set X is an adjacent code, thus $k=0$ and $y'_0=y_0$. Since we have assumed that $|y_0|>|y'_0|$, this case is impossible.
- Assume now that $y_0\in C$. From Lemma 5.4, we have $y'_0y'_1\ldots y'_k=y_0$ and $z=\varepsilon$ since $y'_0\neq\varepsilon$. By definition of C, $y_0\notin X^*$, thus there exists a smallest i, $0\leqslant i\leqslant k$, such that $y_i\in C$. If $i\neq 0$ then $y_0\in X^+t_1A^*$ which is in contradiction with $y_0\in C$. Thus we have $i=0$. In this case, if there exists a smallest j, $0<j\leqslant k$, such that $y'_j\in C$ then $y_0\in A^*t_2X^*t_1A^*$ which is also in contradiction with $y_0\in C$. Hence, we have $y_j\in X$ for $0<j\leqslant k$. If $k>0$ then $y_0\in A^*t_2X^+$ which is impossible. Therefore $k=0$ and $y_0=y'_0$. Since we have assumed that $|y_0|>|y'_0|$, this case is impossible.

Thus the set Y is a code. □

Proposition 5.7. *The set Y is a f.i.d. code with delay d.*

Proof. Let $c_1,c_2,\ldots,c_d\in Y$, let $y_1,y_2,\ldots,y_n\in Y$, $n\geqslant 0$, let $s\in S(Y)$ and let $p\in P(Y)$ such that the word $c_1c_2\ldots c_d$ has a Y-interpretation $(s,y_1y_2\ldots y_n,p)$. We shall prove that this interpretation is quasi-trivial.

Assume first that $s\in S(X)$. We prove that in this case $s\in X^*$.

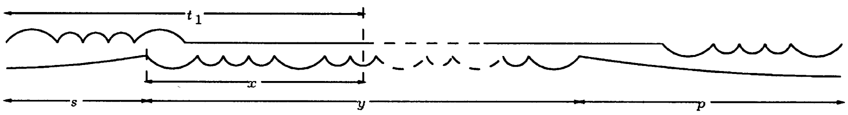

Fig. 7. $s \in S(t_2)$ and $|s| \leqslant |t_2|$.

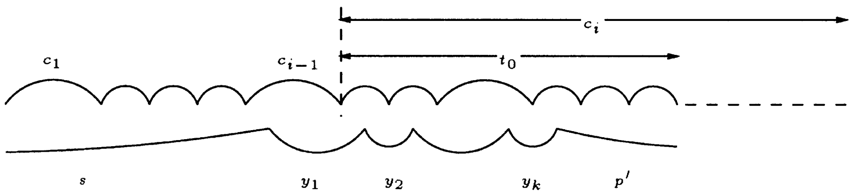

Fig. 8. $s \in S(X)$ and $c_i \in C$.

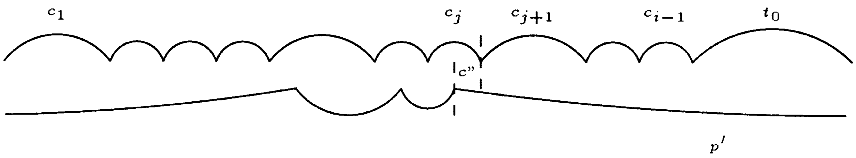

Fig. 9. $p' = c''c_{j+1} \dots c_{i-1}t_0$.

- Assume that there exists a smallest i such that $c_i \in C$ (see Fig. 8).
  Since $t_0 \notin F(X)$, we have $|s| < |c_1c_2 \dots c_{i-1}t_0|$; hence, the word $c_1c_2 \dots c_{i-1}t_0$ has a Y-interpretation $(s, y_1y_2 \dots y_k, p')$ induced by $(s, y_1y_2 \dots y_n, p)$.
  - If this interpretation is an X-interpretation, since $c_1c_2 \dots c_{i-1}t_0 \in X^* \cdot X^d$ and X is a code with delay d, it is quasi-trivial and $s \in X^*$.
  - Assume now that the interpretation is not an X-interpretation. If there exists j such that $y_j \in C$ then the word y_j is not a factor of a word in X. Since we have $c_1c_2 \dots c_{i-1}t_0 \in X^*$, y_j has an X-interpretation induced by the X-factorization of $c_1c_2 \dots c_{i-1}t_0$. Thus, this case is impossible from Corollary 5.5.
    Therefore, $y_1 \dots y_k \in X^*$ and $p' \in P(C)$ since $(s, y_1y_2 \dots y_k, p')$ is not an X-interpretation.
  - If $p' \in P(X^*)$ then $(s, y_1y_2 \dots y_k, p')$ yields an X-interpretation for the word $c_1c_2 \dots c_{i-1}t_0$, therefore it is quasi-trivial, hence $s \in X^*$.
  - Otherwise, this implies that $p' \in t_0A^*$. Let $j \geqslant 1$ and $c'' \in S(c_j)$ such that $p' = c''c_{j+1} \dots c_{i-1}t_0$ (we recall that p' is a suffix of $c_1 \dots c_{i-1}t_0$ and $|p'| > |t_0|$, see Fig. 9).
    The X-interpretation $(c'', c_{j+1} \dots c_{i-1}t_0, \varepsilon)$ of p' induces an X-interpretation (c'', α, β) for the prefix t_0 of p'. This interpretation is quasi-trivial since $t_0 \in X^d \cdot X^*$.

Hence $c'' \in X^*$ and $\beta \in X^*$. We have $t_0 = c''\alpha\beta$ therefore, since X is a code, there exists q such that $t_{0,0} \dots t_{0,q} = c''$.

We have $c_1 \dots c_{i-1} t_0 = s y_1 \dots y_k p' = s y_1 \dots y_k c'' c_{j+1} \dots c_{i-1} t_0$, hence we have $c_1 \dots c_j = s y_1 \dots y_k c''$ thus $c_1 \dots c_j = s y_1 \dots y_k t_{0,0} \dots t_{0,q}$. The word $c_1 \dots c_j$ has an X-interpretation $(s, y_1 \dots y_k t_{0,0} \dots t_{0,q}, \varepsilon)$; since X is adjacent, we have $s \in X^*$.

- Assume now that $c_i \in X$ for $1 \leqslant i \leqslant d$. With Corollary 5.5, we have $y_j \in X$. For $1 \leqslant j \leqslant n$. Moreover, if $p \in P(X^*)$ then we have an X-interpretation for $c_1 \dots c_d$ and thus it is quasi-trivial, hence $s \in X^*$. Otherwise, we have $p \in t_0 A^*$ and the same argument as above in replacing p' by p yields $s \in X^*$.

Assume that $s \in S(C)$. We shall prove that $s \in Y^*$. If $s \in S(X^*)$ then we set $s = s's''$ with $s' \in S(X)$ and $s'' \in X^*$. The previous study leads to $s' \in X^*$, hence $s \in X^*$.

We consider the case where $s \in A^* t_0$.

- If $c_1 \dots c_d$ induces a Y-interpretation $(c', c_i \dots c_{i+j}, c'')$ for the suffix t_0 of s then, since the word c_i belongs to X or C and since the words of C have t_0 as prefix and suffix, this interpretation is an X-interpretation. Therefore, since $t_0 \in X^d \cdot X^*$ and since X has delay d, it is a quasi-trivial interpretation. Hence, since $s = c_1 \dots c_{i+j} c''$, we have $s \in Y^*$.
- If $c_1 \dots c_d$ does not induce a Y-interpretation then t_0 is a factor of a word of $X \cup C$. Since $t_0 \notin F(X)$, the word t_0 is a factor of a word c_i in C. In this case, c_i has a Y-interpretation $((c_1 \dots c_{i-1})^{-1} s, \alpha, \beta)$ induced by $(s, y_1 \dots y_n, p)$. From Lemma 5.4, the interpretation is equal to $(\varepsilon, c_i, \varepsilon)$, $(c_i, \varepsilon, \varepsilon)$ or $(\varepsilon, \varepsilon, c_i)$. Thus we have $s = c_1 \dots c_{i-1}$ or $s = c_1 \dots c_i$, hence $s \in Y^*$.

We have proved that $s \in Y^*$. In a similar way, we have $p \in Y^*$. Thus the Y-interpretation $(s, y_1 \dots y_n, p)$ of $c_1 \dots c_d$ is quasi-trivial. Therefore Y is a f.i.d. code with delay d. $\square$

Proposition 5.8. *The code Y is complete.*

Proof. We shall prove that for any word $w \in A^*$ we have $t_1 w t_2 \in Y^*$.

If $t_1 w t_2 \in X^*$ then $t_1 w t_2 \in Y^*$. Assume that $t_1 w t_2 \notin X^*$.

Let $x, y \in X^*$ and let $u, v \in A^*$ such that $t_1 w t_2 = x t_1 u = v t_2 y$ with x and y of maximal length. We shall prove that we have $|t_1 w t_2| \geqslant |x t_1| + |t_2 y|$, that is $|x| + |y| \leqslant |w|$.

Indeed, if $|t_1 w t_2| < |x t_1| + |t_2 y|$, then $x t_1$ has a suffix which is a prefix of $t_2 y$, thus we have $|t_1 w t_2| + |t_1| \leqslant |x t_1| + |t_2 y|$ (if $|t_1 w t_2| < |x t_1| + |t_2 y| < |t_1 w t_2| + |t_1|$ then a suffix of t_1 is a prefix of t_2, and Lemma 5.3 ensures that it is impossible), that is $|x| + |y| \geqslant |w| + |t_1|$. Let z be the suffix of $x t_1$, prefix of $t_2 y$, such that $x t_1 = x' z$, $t_2 y = z y'$ and $t_1 w t_2 = x' z y'$. Since $|x| + |y| \geqslant |w| + |t_1|$, we have $|x t_1| + |y t_1| \geqslant |w| + 3|t_1|$, thus $|x'| + |y'| + 2|z| \geqslant |x'| + |z| + |y'| + |t_1|$, therefore $|z| \geqslant |t_1|$. Hence, there exist $\alpha \in S(X^*)$, $\beta \in P(X^*)$ such that $z = \alpha t_1 = t_2 \beta$, $x = x'\alpha$ and $y = \beta y'$ (see Fig. 10).

- If t_1 is a suffix of β, then β induces an X-interpretation for the prefix t_0 of t_1. By definition of t_0, it is a quasi-trivial interpretation. Since $\beta y' \in X^*$, we have $t_1 y' \in X^*$ (see Fig. 11). We have $t_1 w t_2 = x' \alpha t_1 y'$, therefore $t_1 w t_2 \in X^*$. We have assumed $t_1 w t_2 \notin X^*$, thus t_1 is not a suffix of β.

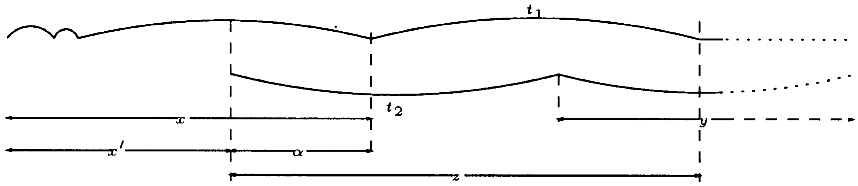

Fig. 10. $|x| + |y| \geqslant |w| + |t_1|$.

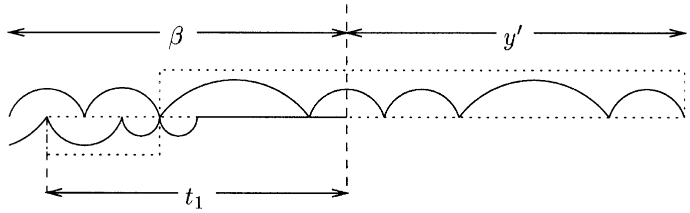

Fig. 11. $\beta y' \in X^*$ and $t_1 y' \in X^*$.

- Hence, β is a suffix of t_1. We have $z = t_2\beta = \alpha t_1$. Thus, there exists $u' \in S(t_2)$ such that $t_1 = u'\beta$. With Lemma 5.3 (ii), $u' \in X^*$. We have $t_1 w t_2 = x'\alpha t_1 y' = x'\alpha \cdot u' \cdot \beta y'$, thus $t_1 w t_2 \in X^*$. This is in contradiction with the hypothesis.

Therefore, we have $|t_1 w t_2| \geqslant |x t_1| + |t_2 y|$. Let r be the word such that $w = x t_1 r t_2 y$. Note that $t_1 r t_2 \notin X^*$ since $t_1 w t_2 \notin X^*$ and $t_1 r t_2 \notin X^+ t_1 A^* \cup A^* t_2 X^+$ by maximality of $|x|$ and $|y|$.

- If $t_1 r t_2 \notin A^* t_2 X^* t_1 A^*$ then we have $t_1 r t_2 \in C$. Hence $t_1 w t_2 \in Y^*$.
- If $t_1 r t_2 \in A^* t_2 X^* t_1 A^*$. Let r_1 be the word such that $r \in r_1 t_2 X^* t_1 A^*$ with r_1 of minimal length (we recall that a prefix of t_2 cannot be a suffix of t_1). We shall prove that $t_1 r_1 t_2 \in C$. By definition of r_1, we have $t_1 r_1 t_2 \notin A^* t_2 X^* t_1 A^*$. In a similar way, we have $r_1 \notin A^* t_2 X^+$. Since x is of maximal length, we have $t_1 r_1 t_2 \notin X^+ t_1 A^*$. Let r_2 be the word such that $r \in r_1 t_2 X^* t_1 r_2$ with r_2 of minimal length. By induction on $t_1 r_2 t_2$ we have $t_1 w t_2 \in Y^*$.

Thus the set Y is complete. □

Proposition 5.9. *The code Y is very thin.*

Proof. We shall prove that $t_1 t_2 t_1 t_2 \in Y^* \backslash F(Y)$. Due to the above proof, we have $t_1 t_2 \in Y^*$, hence $t_1 t_2 t_1 t_2 \in Y^*$. Moreover, $t_1 t_2 t_1 t_2 \notin F(X)$ since $t_0 \notin F(X)$ and $t_1 t_2 t_1 t_2 \notin F(C)$ since $t_1 t_2 t_1 t_2 \in A^* t_2 t_1 A^*$. □

Remark 5.1. As for circular codes (see [2, p. 328]), finite f.i.d. codes different from A cannot be embedded into a finite maximal code. However, the set Y constructed in Theorem 5.2 still remains regular when X is regular.

Finally, note that owing to Remark 4.1 we are able to embed any thin f.i.d. code in a complete very thin f.i.d. code with the same delay.

Theorem 5.10. *Let X be a f.i.d. code with delay d. Then X can be embedded into a complete f.i.d. code Y with delay d. Moreover if X is rational, then Y is rational.*

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